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An Approximate *LU* Factorization Method for the Compressible Navier–Stokes Equations

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The approximate *LU* factorization is applied to the Beam–Warming–Steger method for the compressible Navier–Stokes equations. This factorization is mainly based on the implicit flux vector splitting technique to which the simple estimation of eigenvalue of the viscous terms is added. The two-dimensional interaction problem of shock wave with laminar boundary layer on a flat plate was solved. Numerical results confirm the efficiency and reliability of the present method.

1. INTRODUCTION

The compressible Navier–Stokes equations have been used in computational fluid dynamics only for a limited number of special researches because of a large amount of computational time and computer storage. Development of efficient methods for the compressible Navier–Stokes equations is desired.

Various factorization or splitting of the implicit procedure of finite-difference methods in delta-form can be applied to improve convergence rates for steady-state problems. Some attempts for the Euler equations have been made by employing the local eigensystem [1], such as the diagonal form [2] and the flux vector splitting [3]. The use of the local eigensystem implies an implicit upwind difference algorithm where only a lower or upper bidiagonal matrix appears and the inversion is easier than that of tridiagonal matrix. Such algorithm has an advantage to stability and efficiency. For the compressible Navier–Stokes equations, the implicit MacCormack method [4] retains its efficiency due to the simple estimation of the viscous terms added to the implicit procedure for the stability, although the steady-state solutions depend on a time increment.

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An upwind difference implies a central difference and an additional smoothing term in principle. The use of an upwind one in the explicit part results in the finite-difference methods with uncontrollable smoothing terms. A combination of a central difference and an adequate smoothing term originates accuracy and reliability required for a steady state.

In this paper, an approximate LU factorization is proposed by applying it to the Beam–Warming–Steger algorithm. Each ADI operator is decomposed into the product of the lower and upper bidiagonal matrices by the LU factorization derived from the ideas of the flux vector splitting and the implicit MacCormack method. This LU–ADI factorization leads to the inversions of the scalar bidiagonal matrices with the diagonalization technique. The resulting method reduces CPU time and temporay storage, and retains the second-order accuracy in space and the reliability for steady-state solutions.

2. Algorithm Development

A. *Basic Algorithm*

The governing equations of compressible viscous fluids are the compressible Navier–Stokes equations. They are written in the conservation-law form. For brevity, a one-dimensional equation of conservation laws is at first considered to construct the new scheme and to analize its stability. The resulting scheme is then extended to the compressible Navier–Stokes equations in the following section.

A model equation is written as,

$$u_t + f_x = 0, \tag{1}$$

where u is the density, f is its flux, and the subscripts t and x denote partial differentiations. This system can be rewritten as,

$$u_t + a(u)\, u_x = 0, \tag{2}$$

where $a = \partial f/\partial u$, if $f = f(u)$. The standard implicit finite-difference method in delta-form can be employed to seek steady-state solutions:

$$(1 + \theta h D a)\, \Delta u_i^n = -hD(f_i^n), \tag{3}$$

where $u_i^n = u(i\Delta x, n\Delta t)$, $D(f_i) = f_{i+1/2} - f_{i-1/2}$, $Da\Delta u_i^n = D(a\Delta u_i^n)$, $h = \Delta t/\Delta x$ and θ is a number between 0 and 1.

First, let f be a function of u. This case implies the Euler equations. The flux f on half mesh-points are evaluated as,

$$f_{i\pm 1/2} = (f_i + f_{i\pm 1})/2. \tag{4}$$

Therefore the difference operator $D(f_i)$ results in the second-order central difference for first derivatives; $D(f_i)=(f_{i+1}-f_{i-1})/2$. The linearized coefficients $a_{i\pm 1/2}$ can be chosen [5] as

$$a_{i\pm 1/2}=c_{i\pm 1/2}=\left(\frac{\partial f}{\partial u}\right)^n_{i\pm 1/2}. \tag{5}$$

The central finite-difference can be replaced by the sum of the upwind differences, using the flux vector splitting technique [3]:

$$D(c\Delta u)=D_-(c^+\Delta u)+D_+(c^-\Delta u), \tag{6}$$

where $c^+=(c+|c|)/2$, $c^-=(c-|c|)/2$, $D^-(c^+\Delta u_i)=c^+\Delta u_i-c^+\Delta u_{i-1}$ and $D^+(c^-\Delta u_i)=c^-\Delta u_{i+1}-c^-\Delta u_i$.

The modified implicit procedure of Eq. (3) is rewritten as,

$$(1+\theta h(D_-c^+ +D_+c^-))\,\Delta u_i^n=-hD(f_i^n). \tag{7}$$

The usual Neumann's stability analysis for constant coefficients results in the condition of θ such that $\theta \geqslant 1/2$. This scheme is second-order accurate when $\theta=\frac{1}{2}$. The first-order scheme, when $\theta=1$, is adequate to calculate a steady-state solution and thus used here. The left-hand side of Eq. (7) can be replaced by the LU factored form suggested in Ref. 3, because $h^2=O(\Delta t^2)$.

$$(1+hD_-c^+)(1+hD_+c^-)\,\Delta u_i^n=-hD(f_i^n). \tag{8}$$

For a linearized analysis, this factorization introduces no error since $c^+c^-=0$. The above operators, $(1+hD_-c^+)$ and $(1+hD_+c^-)$, lead to the lower and upper triangular matrices, repsectively. This form requires no inversion of tridiagonal matrices.

Next, let f be a linear function of u_x, that is, $f=-\mu u_x$ where $\mu>0$ is the viscosity. This case implies the diffusion equations. The values of the flux f on half mesh points are calculated by the following equations,

$$f_{i\pm 1/2}=f(D_\pm(u_i)/\Delta x). \tag{9}$$

In this case, $a_{i\pm 1/2}$ describe the difference operators [6], $-\mu D_\pm(*)/\Delta x$, because,

$$\Delta f^n_{i\pm 1/2}=-\mu D_\pm(\Delta u_i^n)/\Delta x+O(\Delta t^2). \tag{10}$$

The implicit scheme Eq. (3) is written as,

$$(1+\theta h(\mu D_-/\Delta x-\mu D_+/\Delta x))\,\Delta u_i^n=-hD(f_i^n), \tag{11}$$

where θ must satisfy $\theta \geqslant (2h\mu - \Delta x)/4h\mu$ for the stability and $\theta = 1$ is sufficient for a steady-state solution. Eq. (11) can be written in the LU factored form like Eq. (8),

$$(1 + hkD_-)(1 - hkD_+)\,\Delta u_i^n = -hD(f_i^n), \tag{12}$$

where $k = \mu/\Delta x$.

Finally, let f be the sum of functions of u and u_x, that is, $f(u, u_x) = f_1(u) + f_2(u_x)$. This case implies the Navier–Stokes equations. The evaluations on half mesh-points are described as Eqs. (4) and (9) for f_1 and f_2, respectively. The change of the flux f in time is estimated as follows,

$$\Delta f^n = c\Delta u^n + \mu D_\pm(\Delta u^n)/\Delta x + O(\Delta t^2). \tag{13}$$

Corresponding to Eqs. (7) and (11), Eq. (3) can be rewritten as,

$$(1 + h(D_-(|c^+| + k) - D_+(|c^-| + k)))\,\Delta u_i^n = -hD(f_i^n). \tag{14}$$

The resulting LU factored form is written as,

$$\begin{aligned}(1 + hD_-\hat{a}^+)(1 - hD_+\hat{a}^-)\,\Delta u_i^n &= -hD(f_i^n),\\ \hat{a}^\pm &= |c^\pm| + k.\end{aligned} \tag{15}$$

This implicit procedure is of $O(\Delta t)$, but it does not affect the accuracy of a steady-state solution if it exists uniquely. This scheme is of $O(\Delta x^2)$ at a steady state.

B. *The Navier–Stokes Equations*

The two-dimensional compressible Navier–Stokes equations are written in the conservation-law form,

$$\begin{aligned}
&U_t + F_x + G_y = \mathrm{Re}^{-1}(R_x + S_y),\\
&\quad U = (\rho, \rho u, \rho v, e)^{\mathrm{T}},\\
&\quad F = (\rho u, \rho u^2 + p, \rho uv, u(e+p))^{\mathrm{T}},\\
&\quad G = (\rho v, \rho uv, \rho v^2 + p, v(e+p))^{\mathrm{T}},\\
&\quad R = (0, \tau_{xx}, \tau_{xy}, r)^{\mathrm{T}},\\
&\quad S = (0, \tau_{xy}, \tau_{yy}, s)^{\mathrm{T}},\\
&\quad \tau_{xx} = (\lambda + 2\mu)\,u_x + \lambda v_y, \qquad \tau_{xy} = \mu(u_y + v_x), \qquad \tau_{yy} = (\lambda + 2\mu)\,v_y + \lambda u_x,\\
&\quad r = u\tau_{xx} + v\tau_{xy} + \alpha(c^2)_x, \qquad s = u\tau_{xy} + v\tau_{yy} + \alpha(c^2)_y,\\
&\quad \alpha = \mu/\mathrm{Pr}(\gamma - 1).
\end{aligned} \tag{16}$$

The superscript T denotes transpose of a vector. A perfect and calorically perfect gas is assumed as follows,

$$p=(\gamma-1)(e-\rho(u^2+v^2)/2), \qquad c^2=\gamma p/\rho. \tag{17}$$

The Beam–Warming–Steger method [5, 6] applied to Eq. (16) results in the following approximate factorization,

$$\begin{gathered}(I+hD_i(A+P))(I+hD_j(B+Q))\,\Delta U_{ij}^n=-hLr_{ij}^n,\\ Lr_{ij}^n=D_i(F_{ij}^n-R_{ij}^n/\mathrm{Re})+D_j(G_{ij}^n-S_{ij}^n/\mathrm{Re}),\\ A=\left(\frac{\partial F}{\partial U}\right)_{ij}^n, \qquad B=\left(\frac{\partial G}{\partial U}\right)_{ij}^n,\end{gathered} \tag{18}$$

where $U_{ij}^n=U(i\Delta x, j\Delta y, n\Delta t)$, D_i and D_j are the difference operator for i and j, respectively, and $h=\Delta t/\Delta x=\Delta t/\Delta y$. The viscous terms R and S are linearized as follows [6]. The elements of R are of the general form: $f_m=a_m\partial b_m/\partial x$. Each element linearizes in time:

$$\Delta f_m^n=a_m\frac{\partial}{\partial x}\left(\sum_{m'}\frac{\partial b_m}{\partial q_{m'}}\Delta q_{m'}^n\right), \tag{19}$$

where q_m indicates the element of U and it is assumed that $\partial a_m/\partial q_m=0$. This algorithm requires inversions of the block-tridiagonal matrices composed of the block matrices A, B, P and Q.

The Jacobian matrices A and B are diagonalized [1] as,

$$\begin{gathered}A=XE_AX^{-1}, \qquad B=YE_BY^{-1},\\ E_A=\mathrm{diag}(u, u, u+c, u-c) \quad \text{and} \quad E_B=\mathrm{diag}(v, v, v+c, v-c),\end{gathered} \tag{20}$$

where X and Y are the eigenvector matrices. The diagonal matrices E_A and E_B can be split along the sign of each eigenvalue;

$$E_A=E_A^+ + E_A^- \quad \text{and} \quad E_B=E_B^+ + E_B^-. \tag{21}$$

The operators in Eq. (18) can be replaced by using the flux vector splitting [3], for example,

$$D_iA=D_{i-}XE_A^+X^{-1}+D_{i+}XE_A^-X^{-1}. \tag{22}$$

There are two ways to extend the LU factorization to the two-dimensional method. One is the original LU factored form proposed by Jameson and Turkel [7]. The other is that applied to the standard ADI scheme. The formar requires inversions of block matrices. On the other hand, the latter can be described without inversion of a block matrix similarly to the procedure of the implicit MacCormack method. The resulting algorithm is efficient and thus proposed here.

The LU factored form can be obtained as,

$$I + hD_i(A + P) = (I + hD_{i-}(XE_A^+ X^{-1} + \hat{P}))(I + hD_{i+}(XE_A^- X^{-1} - \hat{P})), \quad (23)$$

if $D_i P$ can be rewritten as $D_{i-}\hat{P} - D_{i+}\hat{P}$. The eigenvalues of the block matrix $\hat{P}$ are related to the stability for the discretized viscous terms. On the other hand, smooting terms can be added to the flux vector as a weight of upwind differences [3];

$$\hat{E}_A^{\pm} = |E_A^{\pm}| + kI. \quad (24)$$

The parameter k can be chosen so as to maintain the stability of the viscous terms as follows,

$$k = \frac{v}{\mathrm{Re}\,\rho \Delta x}, \quad v = \max\left(2\mu, \lambda + 2\mu, \frac{\gamma\mu}{\mathrm{Pr}}\right). \quad (25)$$

This estimation is similar to the implicit MacCormack scheme, and v can be identically set to 2μ if $\gamma = 1.4$ and $\mathrm{Pr} = 0.7$.

Finally, the LU factored scheme is described as,

$$(I + hD_{i-}\hat{A}^+)(I - hD_{i+}\hat{A}^-)(I + hD_{j-}\hat{B}^+)(I - hD_{j+}\hat{B}^-)\,\Delta U_{ij}^n = -hLr_{ij}^n,$$
$$\hat{A}^{\pm} = X(|E_A^{\pm}| + kI)\,X^{-1}, \qquad \hat{B}^{\pm} = Y(|E_B^{\pm}| + kI)\,Y^{-1}, \quad (26)$$
$$k = \frac{2\mu}{\mathrm{Re}\,\rho\Delta x} = \frac{2\mu}{\mathrm{Re}\,\rho\Delta y},$$

where the absolute value of a matrix is defined as the matrix whose elements are replaced by their absolute values.

The usual fourth-order dissipation [5, 6] is added to the right-hand side of Eq. (26). On the other hand, the implicit smoothing terms are not required in the following test problem.

The resulting scheme is of $O(\Delta t, \Delta x^2)$ and unconditionally stable for linearized analysis. It is efficient, and needs less temporary storage because inversion of block-tridiagonal matrix is not required. It is also easy to program and vectorize because the only implicit operators of the Beam–Warming–Steger method should be rewritten and the resulting procedure is similar to the implicit MacCarmack one.

3. Numerical Experiment and Results

A. *Two-Dimensional Test Problem*

The interaction problem of a shock wave with a laminar boundary layer on a flat plate [8–13] was solved as a test for the present scheme. The typical feature is

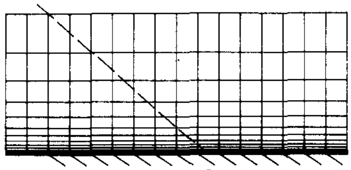

FIG. 1. Computational mesh and incident shock wave. Shock angle $\phi = 32.6°$, Mach number $M = 2.0$, Reynolds' number $Re = 0.296 \times 10^6$; L; the leading edge, S; the incident point of shock wave; (——) typical shock path.

represented by the numerical pressure countour map in Fig. 4d. An oblique shock wave is incident on a laminar boundary layer. The regular reflection imposes pressure gradients on the boundary-layer flow. When the adverse pressure gradient is sufficiently large, the boundary layer separates. The resulting streamline curvature generates reflected shock waves. At the leading edge, a curved bow shock wave appears due to the streamline curvature associated with formation of the boundary layer.

The computational mesh (Fig. 1) at first contained 32×32 mesh-points. The mesh increments were uniform in the x direction as $\Delta x = 1/15$ and exponentially stretched in the y direction as $\Delta y_j = \Delta y_{\min} \times 1.17^j$, where $\Delta y_{\min} = 8.31 \times 10^{-4}$. Mesh-points successively increased by multiples of 32 to 256 points in the x direction and to 96 points in the y direction. The mesh increments decreased, corresponding to mesh refinement. The shock angle was set to 32.6 degrees, the freestream Mach number; 2.0, and the Reynolds number based on the distance from the leading edge to the shock impinging location; 0.296×10^6. Molecular viscosity was calculated by Sutherland's formula. The Prandtl number was set to 0.7 and assumed to be constant. The bulk viscosity for diatomic gas, $\lambda + 2\mu/3$, is known to be neary equal to $2\mu/3$ [14], and thus it is assumed here that $\lambda = 0$. There was no difference between the numerical result under this assumption and that under the Stokes' hypothesis, $\lambda + 2\mu/3 = 0$, in case of 32×32 mesh-points.

The initial condition was taken to be uniform flow. The computation was impulsively started. The incident shock wave was given at the top of the computational region as the fixed boundary conditions. The upstream boundary conditions were also fixed because the freestream was supersonic. The zero-order extrapolation was employed at the boundary of outflow. The reflective boundary conditions were used at the plane of symmetry and at the wall; $U_{1j} = \text{diag}(1, 1, -1, 1)\, U_{2j}$ and $U_{1j} = \text{diag}(1, -1, -1, 1)\, U_{2j}$, respectively.

The implicit boundary condition of ΔU^n at outflow was taken equal to 0. In the y direction, the implicit procedure swept first from the top with $\Delta U^n = 0$ to the wall, then swept to the contrary. The boundary flux of the later sweep was set to 0.

B. *Results and Discussion*

Results for 32×32 and 64×32 mesh-points were obtained when the residual $\|\Delta U\|_2$ reached 10^{-3}, and those for 128×64 and 256×96 mesh-points, when

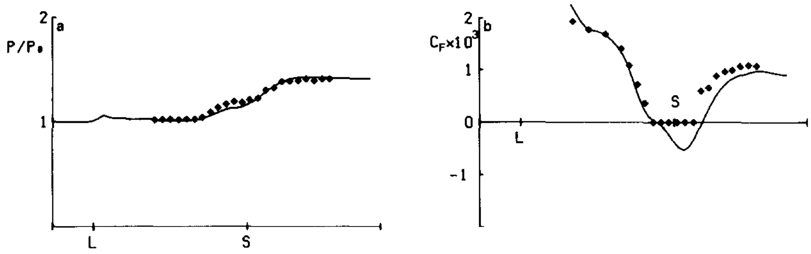

FIG. 2. Numerical results for 64 × 32 mesh-points; (—) present; (◆) experiment; (a) surface pressure, (b) skin friction.

5×10^{-4}. The smoothing coefficient for the former was set to the usual value, Δt. That for the latter was taken equal to $3\Delta t$, since the actual stability lessened in the region of the reflected expansion wave on the boundary layer.

Figures 2a and 2b compare the surface pressure and the skin friction distributions calculated by the present scheme for 64 × 32 mesh-points with those by experiment [12]. Figures 3a and 3b show those for 256 × 96 mesh-points. In the pressure distribution, the peak at the leading edge and the plateau at the separated region are clearly observed. The numerical pressure contours for 32 × 32 to 256 × 96 mesh-points were presented in Figs. 4a–d where the lines indicate the contours of values of 1.05 to 1.40 at intervals of 0.05. Shock waves appeared as a near discontinuity owing to mesh refinement, such as the bow shock wave and the double reflected shock waves. Figure 5 shows various numerical contour maps near the location of incidence for 256 × 96 mesh-points. In Fig. 5c, incomressibility was assumed to evaluate stream function which was set to 0 at the wall and multiplied by 10^2. The

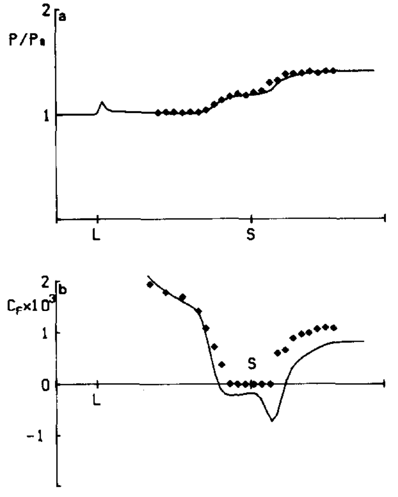

FIG. 3. Numerical result for 256 × 96 mesh-points; (—) present; (◆) experiment; (a) surface pressure, (b) skin friction.

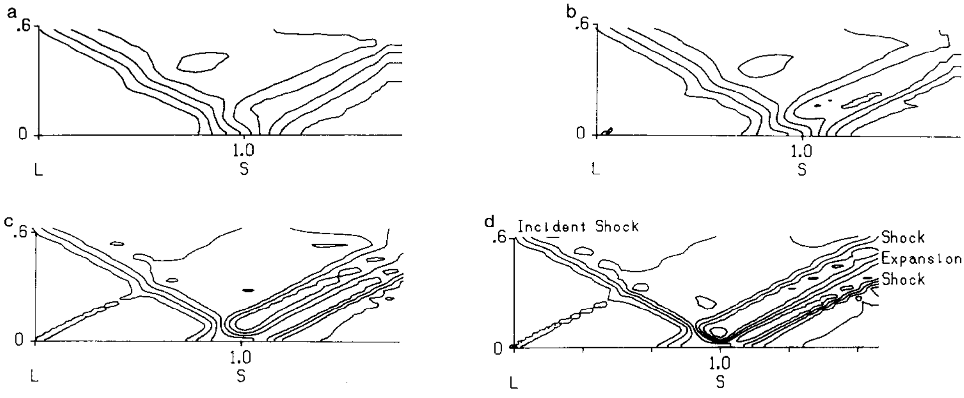

FIG. 4. Numerical pressure contours. (a) 32 × 32 mesh-points, (b) 64 × 32 mesh-points, (c) 128 × 64 mesh-points, (d) 256 × 96 mesh-points.

skin friction distribution in Fig. 3b suggests a structure of the separated bubble, and this is consistent with that obtained in Fig. 5c. These results are consistent with experimental and other numerical results.

It was confirmed that the present scheme has the advantageous property of independence of the size of the time increment owing to the delta-form.

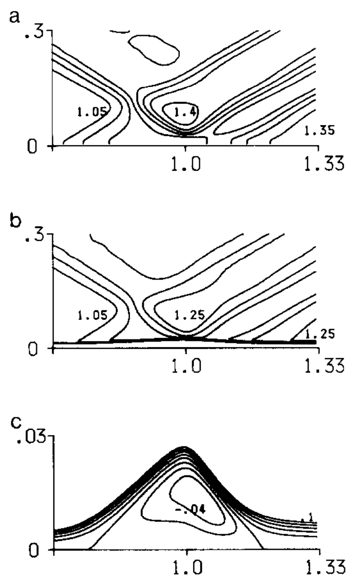

FIG. 5. Numerical contour maps near the incident point of shock wave for 256 × 96 mesh-points. (a) pressure contours at intervals of 0.05, (b) density contours at intervals of 0.05, (c) stream function contours at intervals of 0.02.

The maximum CFL numbers were about 60 in the present computations. For the two finer meshes, the run times required to reach the steady state were about 15 in nondimensional time. The convergence rates in mesh refinement decreased owing to the time increments bounded by the actual stability.

The CPU time per step is reduced about 20% compared with that of the implicit MacCormack method. The computational efforts are totally saved less than a half. This reduction is due to its having less arithmetic operations for the explicit part and to the independence of the size of the time increment.

The CPU time is also reduced about 5% compared with that of the *LU* factored form for the two-dimensional implicit operator proposed by Jameson and Turkel [7, 15], in case of the Euler equations. The *LU* factorization method with ADI is more efficient than that without ADI. The latter requires the inversions of 4×4 block matrices. Such inversions are still costly, although the number of the resulting implicit operators is fewer than that of the present one.

4. Conclusion

The approximate *LU* factorization to the Beam–Warming–Steger method has been developed to efficiently compute steady-state solutions. The resulting method is of $O(\Delta t, \Delta x^2)$ and has the following advantageous properties: 1. efficiency owing to the highly convergence rates and to the reduction of arithmetic operations per time-step, 2. reliability for steady-state solutions. It is also easy to program and vectorize.

The two-dimensional interaction problem of shock wave with laminar boundary layer is solved in order to test the present method. The numerical solutions rapidly reach the steady state with large CFL numbers. They are also independent of time increments. It is shown that the results for the finer mesh precisely represent the global flow field, especially the phenomenon of reflection of the incident shock wave.

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